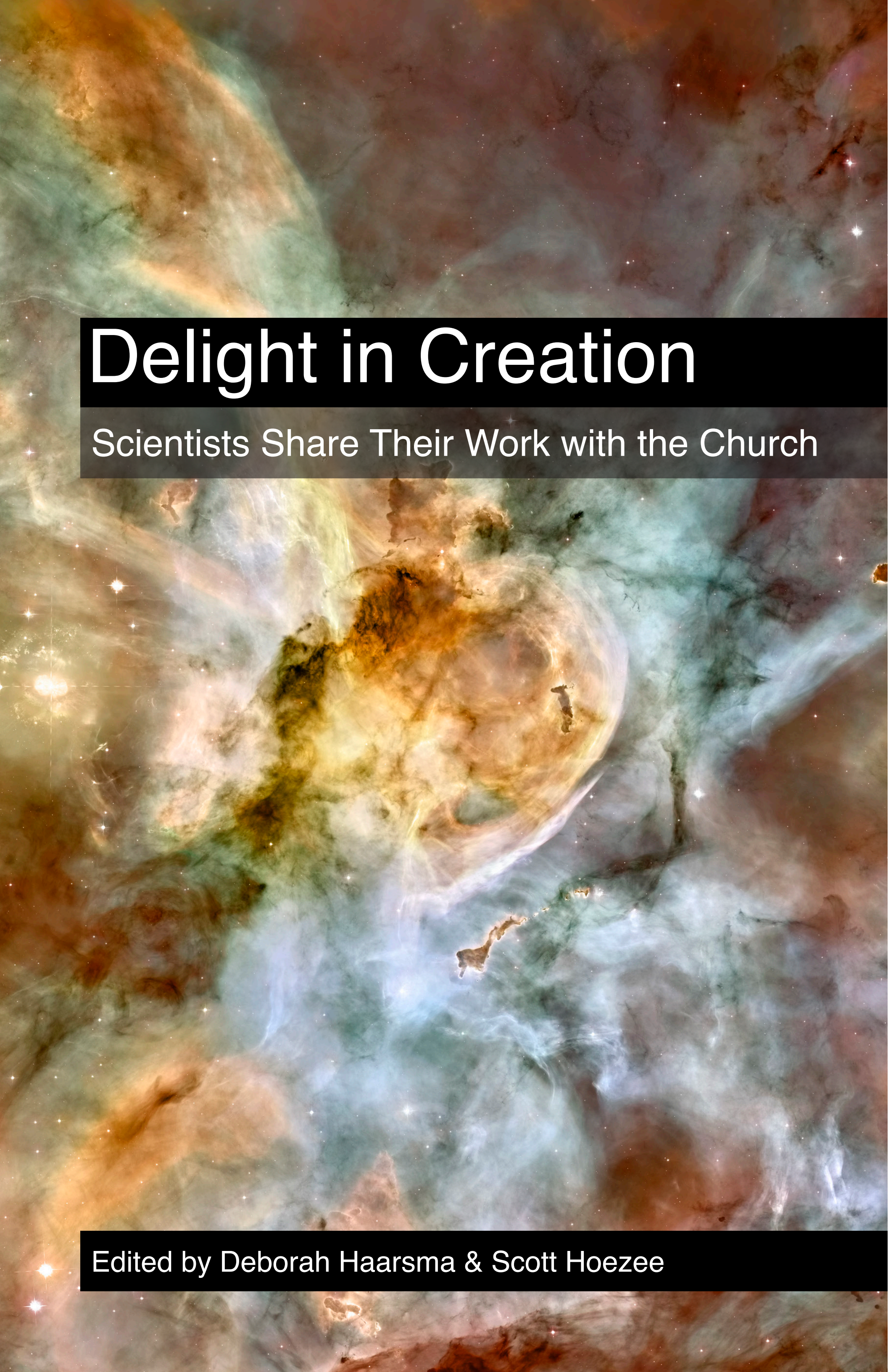

10 Mathematics and Beauty

by James Turner

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Great is the LORD and most worthy of praise;
his greatness no one can fathom.
One generation commends your works to another;
they tell of your mighty acts.
They speak of the glorious splendor of your majesty—
and I will meditate on your wonderful works.
Psalm 145:3-5

Other Worlds

In high school, I was a voracious reader, particularly of science fiction and fantasy novels. It was a form of escapism for me. To imagine myself as a knight in medieval times or a space explorer on an unknown planet was my favorite way of passing the time—so much so that if my studies failed to feed my imagination in the same way, they often took second stage.

During my sophomore year, I encountered a genre of science fiction, written in particular by H.P. Lovecraft, that portrayed our everyday experiences as a mere façade. Behind the façade lay deep, ancient mysteries and forces, often nihilistic, that might someday make their appearance to the detriment of humankind. This type of literature fascinated for me for two reasons. First, much of the writing occurred during an era when research in physics was providing startling new insights into the nature of our physical world. Einstein's theories of special and general relativity were beginning to give us a picture of large-scale space and time as curved rather than flat, as previously thought. Quantum mechanics was illuminating the very small as exhibiting weird physical properties in which particles exist as energy packets, called quanta, which "move" in a discrete fashion. This view blurred our ability to make observable predictions of both their position and velocity. Ideas like higher dimensions and non-Euclidean geometries provided fertile ground for imaginative stories. Often these stories contained descriptions of strange alien races who achieved bewildering advances in science and technology which we would begin to understand only when our mathematics and science one day reached a suitable maturity, perhaps millennia from now.

The second reason these writings fascinated me was that they often found a way to convey a deep sense of wonder and mystery toward the unknown of our world, while portraying science and mathematics as a way of pulling back the veils that hide these mysteries. While I was reading these stories, my geometry class started to hold more of my attention as I looked for evidence of deeper mysteries in the geometric constructions and deductions discussed in class. During my senior year, as I was considering which colleges and universities to apply to, I looked for opportunities to explore topics like curved space and non-Euclidean geometry in ways that hinted at the possibility of wonder these stories promised. I made it my mission to focus my studies on mathematics and seek out the deeper mysteries that might be hidden with our reality.

When I entered college and embraced Christianity, suddenly mathematics and wonder took on completely new meanings.

From Wonder to Beauty

Where there exists wondrous proportion and primal equality…
Saint Augustine, *On the Trinity,* vi. 10

As is often the case, a college or university can be a transformative place for young minds. For me, it was the place where my academic studies fed my imagination rather than being a distraction from the fantasies I sought to indulge. In my study of mathematics, I found hints of the wondrous worlds that the stories I read in high school had suggested, a sense of something that transcended our everyday experience.

In my first year, I encountered something else that also pointed to transcendence beyond our world. I made friends whose lives demonstrated connection with a divine Creator who sought a deep relationship with men and women and initiated that relationship through a historical act of incarnation. The faith that they shared with me opened my eyes to a true source of transcendence: a God who divinely created our reality and fills it with wonder and beauty, a God who provided the true way to knowing him through the salvific work of his Son, as incarnated in Jesus Christ.

Suddenly, I understood our reality and the potential realities mathematics spoke of as being illuminated by the same light, the light of Christ.

Now, what can be said of these wonders in mathematics that mesmerized me? Are they the same things that attract other people to the study of mathematics? At the root of every subject in mathematics are both a sense of quantity and a sense of relations: geometry explores spatial relations, number theory explores natural number relations, analysis studies relations within continuous quantities, and so on. Within each one of these subjects, mathematicians can unlock the mysteries of deep and elegant patterns. From these patterns, mathematicians develop sophisticated theories that expand the context in which these patterns can be found. These theories in turn provide a broader range of possibilities for applications within mathematics and, possibly, within other sciences. It is this process of pattern exploration, theory building, and application which drives the development of the various subjects in mathematics. Within that process, mathematicians find points that can instill inspiration and wonder. Such points have certain features in common which individually or collectively can be said to portray a sense of *beauty*. Here are some of those features:

- **Unexpected connections:** In a study of one or more mathematical subjects, two or more seemingly disparate objects or relations may suddenly be seen as shades of a single web of relations, providing a sense of unity within or across such subjects.
- **Simplicity:** A mathematical theory aims to explain the logic underlying discovered patterns using basic definitions, intuitive truths, and suitably basic constructions. Within such a framework, deep and unexpected connections are most intensely revealed when relations can be explained with the greatest simplicity, enabling ease both in discerning their hidden truths and in articulating and communicating such patterns to others.
- **Openness to new possibilities and deeper connections:** Dwelling upon singular patterns and merely giving them a simple explanation is often insufficient and can lead dead ends. What can instill a deeper sense of inspiration to mathematicians is to develop a framework within the theory that not only explains those relations and patterns investigated, but allows for previously unseen connections to unfold.

Let us now move beyond these broad stroke descriptions of what mathematicians find beautiful to look at some concrete examples that demonstrate some of the features associated with beauty in mathematics.

Beauty, Proof and Symmetry

> *…those things are said to be beautiful which please when seen.*
> Saint Thomas Aquinas, *Summa Theologiae* I-II, q. 27, a. 1, ad. 3

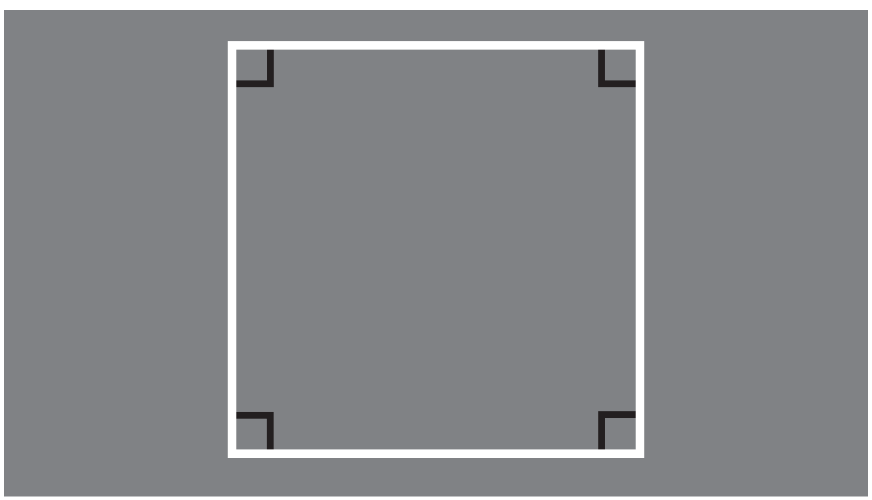
Reflect for a moment on a time when you experienced a sense of wonder or beauty. I would suspect that accompanying that experience was a strong, positive emotion drawing you to attend closer to the object of wonder, to repeat the experience, or perhaps to see if further treasures lay beyond the immediate object. This experience of beauty is what is at play in the mathematician's encounter with mathematics, one which conveys a sense of unveiling the mysteries within the forms and patterns being discerned and deciphered. The type of beauty that is found within a mathematician's world creates a magnetic attraction that pulls upon the mathematician's attention, focusing his or her full being and bringing pleasure in all of the ways the object of beauty is viewed, grasped, and sensed.

As a concrete case study in mathematical beauty, we will concentrate on a subject that often provides the easiest gateway to mathematics through the senses, namely geometry. This ancient form of rigorous mathematics helps us encounter beauty in two particular ways, through rigorous proof and symmetry. We will spend more time with the latter, but I should note that rigorous proof is an important source of aesthetic encounters for the mathematician. For example, consider the following observation from Sir Bertrand Russell in his autobiography: "At the age of eleven, I began

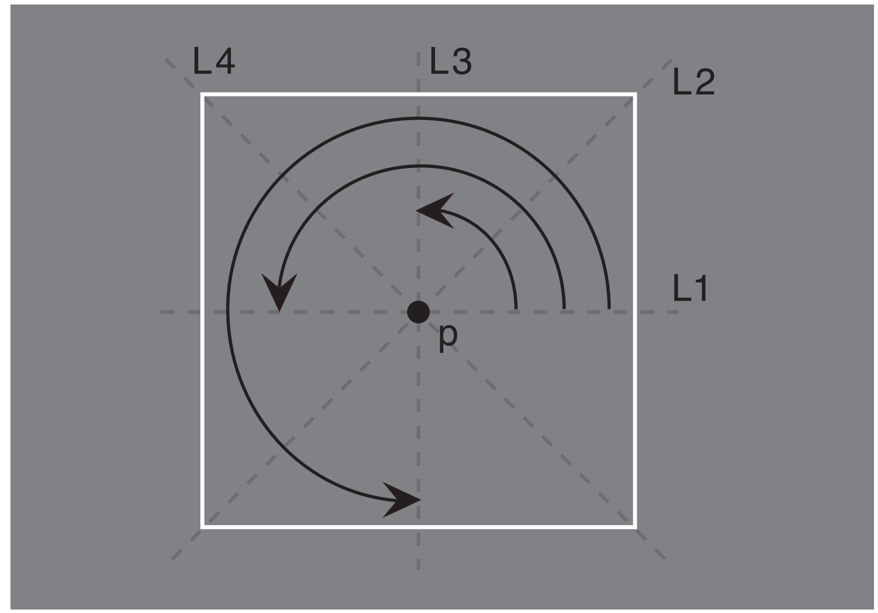
Euclid, with my brother as my tutor. This was one of the great events of my life, as dazzling as first love. I had not imagined that there was anything so delicious in the world." Even more poetic and succinct are the words of Edna St. Vincent Millay: "Euclid alone has looked on Beauty bare."

What is it in the structure of Euclid's *Elements* that evokes such responses of wonder in those who take it upon themselves to study this work? Euclid, in putting together the *Elements*, produced a paradigm for organizing a body of knowledge. Beginning with basic key geometric definitions and self-evident postulates which govern the way these geometric notions relate, the theory of plane geometry is developed from the basic to the more sophisticated through further definitions and constructions, with deeper relations disclosed in propositions. These in turn are established through rigorous proofs that unveil how preexisting relations can be woven together through logic in order to arrive at the desired result. Euclid's approach to conveying mathematical knowledge in his *Elements* has been considered the ideal approach to organizing any subject of mathematics and persists today. It thus provides the means to find beauty in all parts of mathematics in ways that resonate with Russell, St. Vincent Millay, and many others.

Now, let's consider an approach to geometry in which the notion of beauty presents itself in a more common sense fashion, namely through the notion of symmetry. The idea behind symmetry is rather simple. Consider the following square:

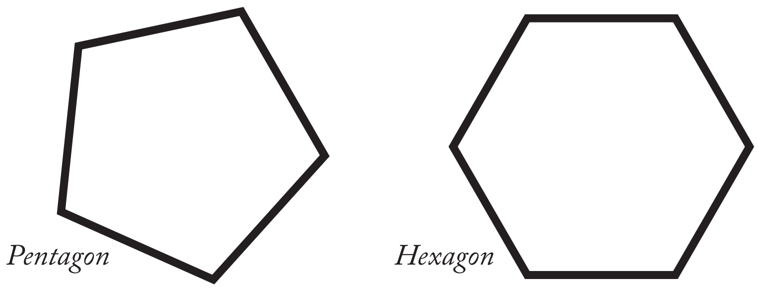


Symmetries of this square can be imagined in two different ways. First, imagine rotating the square counterclockwise through a fixed angle. A rotation is a symmetry if the square looks the same after rotation, such that the vertices and edges have moved to the positions of other vertices and edges. If you can imagine this, you should conclude that the rotations through 0, 90, 180, and 270 degrees give all the useful rotational symmetries. Below is a square with some of these rotations rendered:



The second type of symmetry is a reflection, as illustrated in this same diagram. Fixing your attention upon any one of the four lines L1, L2, L3, and L4 crossing through the center point of the square, imagine spinning the square in space around that fixed line until it lies back in the plane. This gives a reflective symmetry. It is a geometric result that all planar symmetries of the square that fix the center point are either one of the four rotations or one of the four reflections, totaling eight possible planar symmetries. Furthermore, a given symmetry can be related to another symmetry through some third symmetry by a method of composition in which performing one symmetry then applying a second symmetry will result in a third symmetry. For example, rotating the square counterclockwise 90 degrees then reflecting through a horizontal line L1 gives a symmetry that is identical to reflecting the square around the diagonal axis L2. Collectively these symmetries, together with this method of composition, form an example of a structure called a *group*.

Now an analysis of symmetries can be carried out for any geometric shape, not just the square. For example, any polygon in the plane, like the regular pentagon and hexagon below, has symmetries:



The regular pentagon has five rotations and five reflections in its group of symmetries, the regular hexagon has six rotations and six reflections in its group of symmetries, and so on.

If we consider now geometric objects in three spatial dimensions, the analog of regular polygons are the *regular polyhedra*, also known as the *Platonic solids*. In contrast to regular polygons, in which their number is infinite, there are exactly five Platonic solids, as shown on the next page. Each of these solids carries a group of spatial symmetries as well. Recall that for planar objects, the rotations and reflections were about lines, but for these solid objects the rotations and reflections are through planes. As a source of inspired beauty, many mathematicians, philosophers, and scientists, such as Euclid, Plato, and Kepler, have found such deep aesthetic pleasure in the Platonic solids that they've sought to make them building blocks of the universe. For example, Euclid's *Elements* concludes with characterizations and a complete classification of the Platonic solids and Kepler, in his *Mysterium Cosmographicum,* gave a model rendering the solar

Platonic solids.

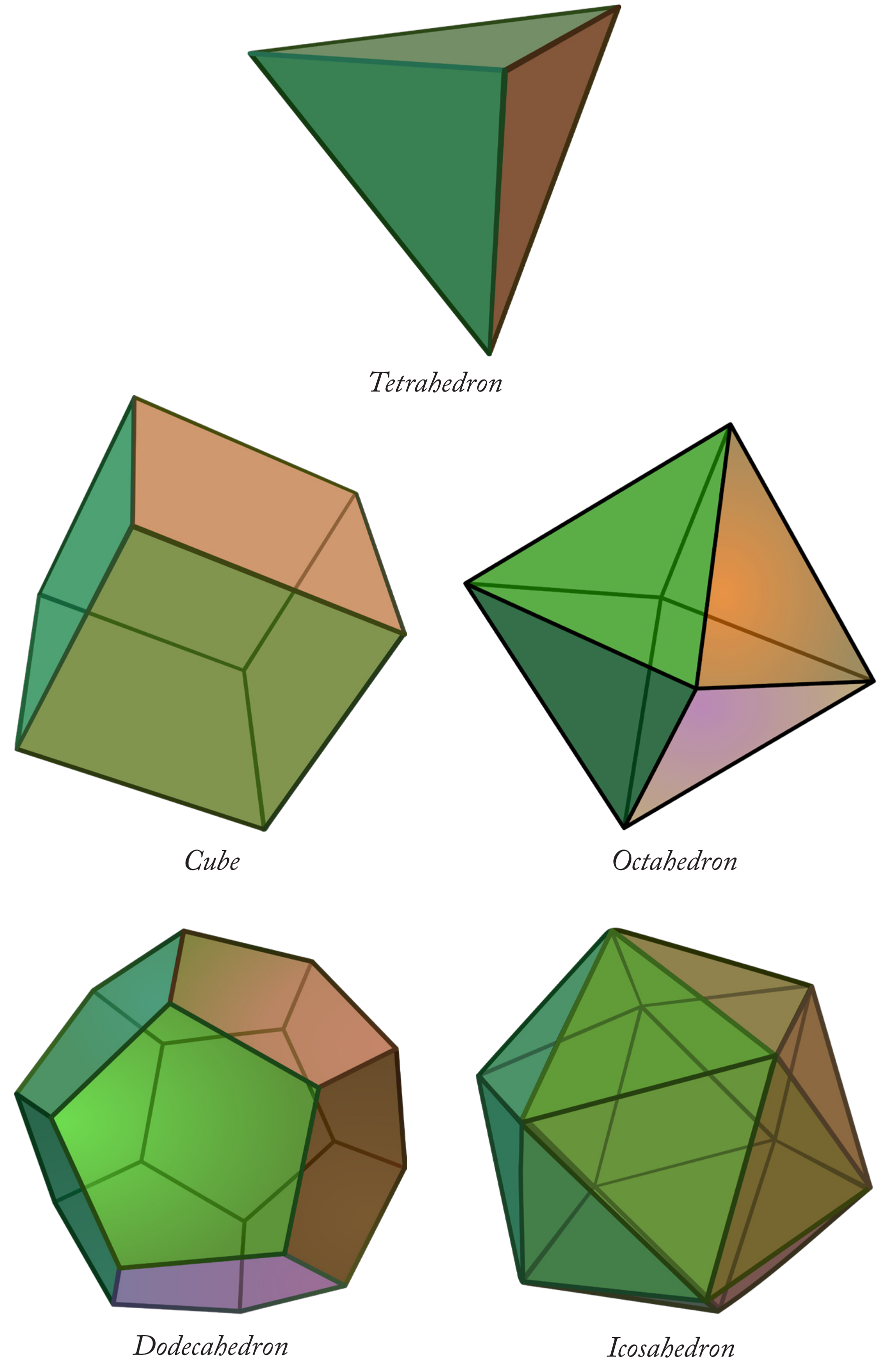


IMAGE CREDIT: WIKIPEDIA COMMONS

system using nested Platonic solids (below). It wasn't until the nineteenth century that mathematicians fully developed the theory of symmetries and their groups.

At the start of the twentieth century, the revolutions in physics—special and general relativity theory and quantum physics—found in the mathematical theory of symmetry the means to model the quantitative properties and relations in the newly understood nature of space and time of the very large or the very small. For example, in Einstein's theory of relativity, his principle of invariance asserts that the same experiment conducted at two different points in space and time will have essentially the same outcome once the appropriate space-time symmetry is taken into account. In quantum physics, the most fundamental of particles possess internal symmetries that individually characterize them as the particles they are. Furthermore, the way these particles interact with each other possess a wealth of symmetries that both characterize the relationships between them and provide the means to locate them experimentally in, for example, particle accelerators. In the current regime of theoretical physics research and exploration, the search for a Grand Unified Theory—a model

Kepler's Mysterium Cosmographicum.

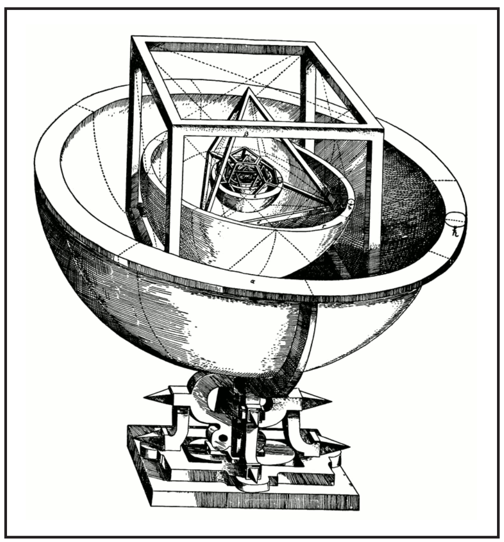


IMAGE CREDIT: WIKIPEDIA COMMONS

of fundamental particles that accounts for all the forces of nature—has led physicists to extend the theory of symmetries of space-time in order to expand our current accounts of particle physics to also include gravity and relativity theory. For example, *superstring theory* incorporates a ginned-up version of symmetry known as *supersymmetry*. Thus, symmetry illustrates all three types of beauty mentioned earlier: unexpected relationships of symmetries into groups, simplicity of the visible geometry of the symmetric relationships, and new possibilities when applied to the physical world.

Exploring the Beyond: Higher Dimensions

Another major advance in mathematics in the nineteenth century that also played a pivotal role in the advances of twentieth century physics was the development of geometry in higher dimensions. Ever since Descartes created analytic geometry in the seventeenth century, Euclid's geometry in the *Elements* could be synthesized with algebra in such a way that mathematicians found rather quickly geometric objects that required more than three spatial dimensions to describe. Moreover, the calculus as created by Newton and Leibniz could be generalized and applied to such geometric objects to enable a deeper understanding of their nature. All of this culminated in the nineteenth century with the work of Bernhard Riemann who developed a general theory of geometry that unified the algebraic and analytic features explored since the seventeenth century. This view of geometry required mathematicians to unshackle their senses in order to comprehend those features of geometric objects that resided in four, or five, or even higher dimensions. Breaking free of our senses in order to comprehend a reality that completely transcends our three-dimensional world requires thinking by analogy to translate our experiences to those of a creature who inhabits such a world of higher dimensions. This development inspired Edwin Abbot Abbot to pen the book *Flatland* in 1884, which describes two-dimensional creatures who live in a planar world and tells the tale of one such creature (a square) who is paid a visit by a three-dimensional creature (a sphere). What unfolds is the attempts of the sphere to explain his nature to square, who can only experience a world of two dimensions. One of Abbot's aims in this novel is to give the reader a window into the then recent exploration by mathematicians into the nature of higher dimensions, which illustrates the idea of transcendence in both its scientific and religious sense.

To see how analogy can give insight into higher dimensions, consider the following sequence of geometric figures:

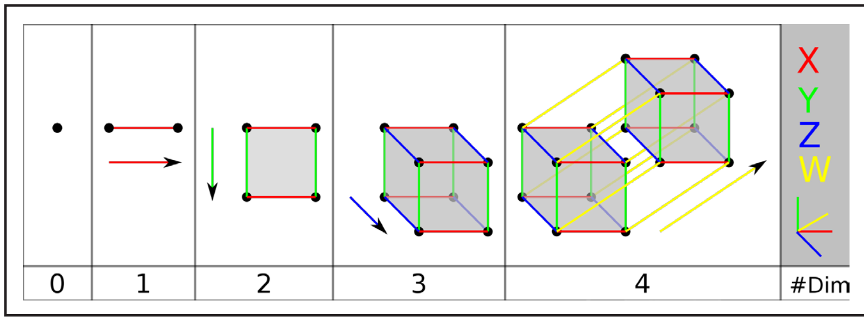


IMAGE CREDIT: VICTORIA SEABURG FOR THE MINISTRY THEOREM, AFTER WIKIPEDIA COMMONS DIAGRAM

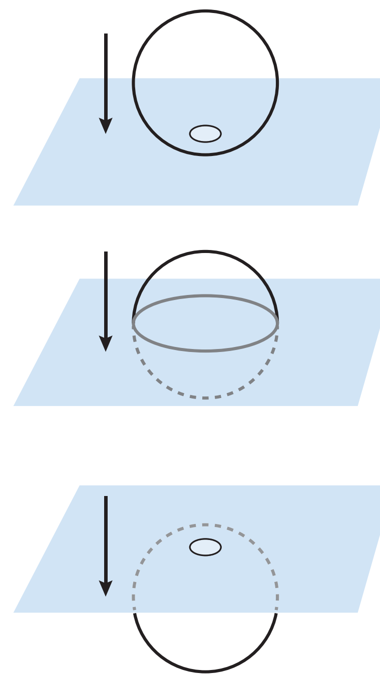
The progression from left to right portrays the notion of cube in the appropriate dimension:

- A point is a zero-dimensional cube.
- A line segment is a one-dimensional cube formed by dragging the point one unit to the right along a one-dimensional axial direction.
- A square is a two-dimensional cube formed by dragging the line segment one unit along a second direction perpendicular to the first axial direction.
- A cube is a three-dimensional cube formed by dragging the square segment one unit along a third axial direction perpendicular to both the first and second axial directions.
- A hypercube (or *tesseract*), then, is a four-dimensional cube formed by dragging the cube segment one unit along a hypothetical fourth axial direction perpendicular to each of the first, second, and third axial directions.

Since we cannot experience such a fourth axial direction, we are left with only a conceptual description about how we may form a hypercube based

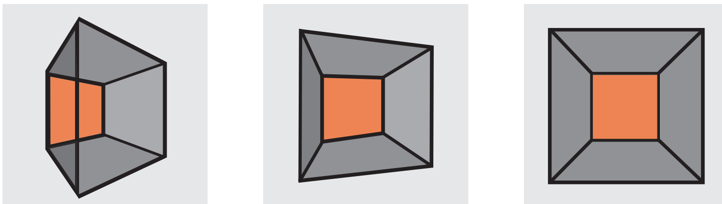
on how lower dimensional cubes are formed. Properties of the hypercube are then extracted through this analogy.

Using analogy to visualize and analyze higher dimensional objects is one of the main ways mathematicians can translate such objects to our realm of experience for study. This process is necessary as visualizing higher dimensional objects in their actual form is extremely difficult. By imagining how such objects can be constructed from lower dimensional objects, as described above, we may surmise the properties of objects in dimensions four, five, and higher in an inductive way by developing objects from dimensions one, two, and three. This is the method of analogy that is so beautifully described by Abbott in *Flatland*, particularly in Square's encounter with the three-dimensional Sphere. As Sphere entered into Flatland, Square perceived a dot which became a circle whose radius grew until it reached Sphere's radial length, and then shrunk back down until the circle reached a point and then disappeared, as illustrated below. By analogy, one way to imagine a hypersphere in four dimensions is by its appearance as it passes through our physical three-dimensional space.



Initially we would see a point and then, like a balloon, we see a small sphere that inflates until it expands to a sphere of radial length equal to that of the hypersphere and then deflates back down to a point.

Another way to visualize the hypercube through analogy is to consider the cube through the following sequence of perspectives:



Here the cube is to be viewed by focusing on the red back face as the viewer comes from the side of the cube, moving until facing the back face directly through the front face. This last perspective of the cube (on the right) can also be viewed in a two-dimensional fashion, as an outer square and an inner square with nearby corners connected by line segments. This way of viewing a cube two-dimensionally is often called a *projection* or *shadow* of the cube. By analogy, we may consider a similar perspective of a hypercube: our three-dimensional view of the cube that focuses on the back face through the front face translates to a four-dimensional view of the hypercube that focuses on the "back cube" through the "front cube." The resulting projection from four-space to three-space translates "inside/outside square" for the two-dimensional projection of a three-dimensional cube to "inside/outside cube" for the three-dimensional projection of a four-dimensional cube and appears as follows:

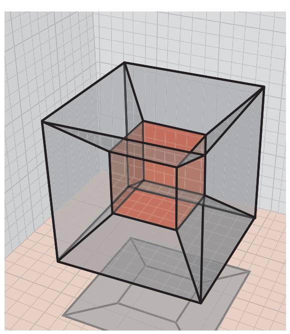


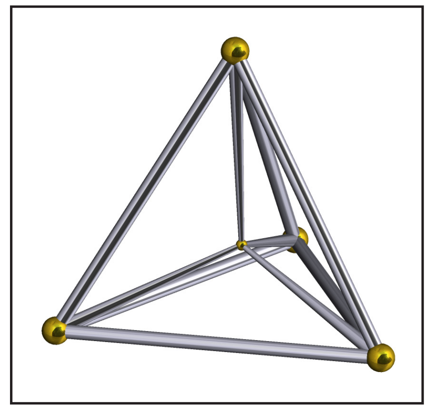
IMAGE CREDIT: VICTORIA SEABURG FOR THE MINISTRY THEOREM, AFTER DREW OLBRICH DIAGRAM

IMAGE CREDIT: WIKIPEDIA COMMONS

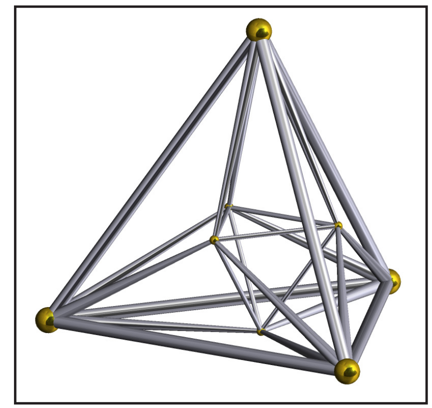
In a similar way, the other Platonic solids possess four-dimensional analogs (see below). Notice the remarkable symmetry and intricacy in the hyper-dodecahedron and the hyper-icosahedron. The simple visual symmetry of these images is a great example of beauty in mathematics.

Now, higher dimensional geometry, as noted before, has found its way into scientists' efforts to understand the universe in the twentieth century through the developments of general relativity theory and quantum physics. One recent way such higher dimensions have entered physical theories is

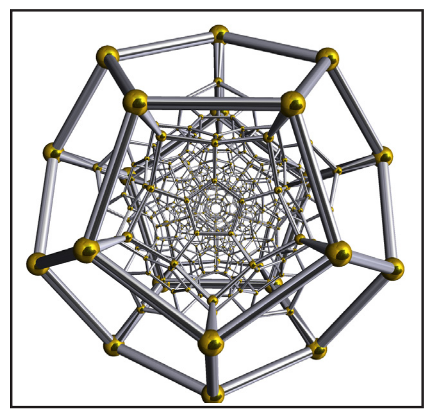
Four dimensional analogs of Platonic solids.



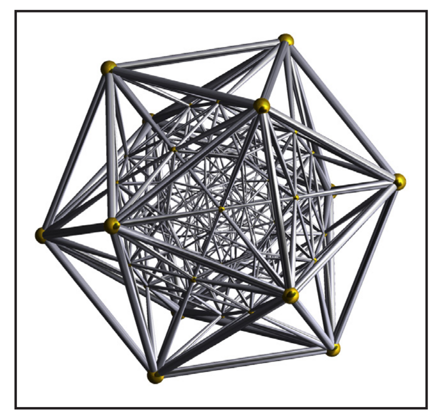
Hyper-tetrahedron



Hyper-octahedron

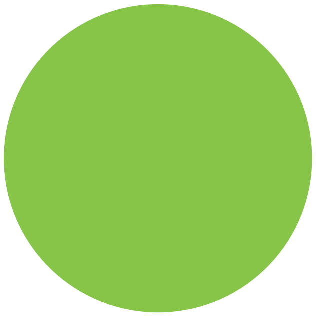


Hyper-dodecahedron



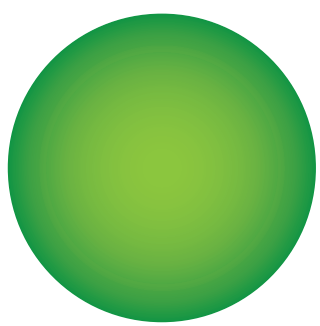
Hyper-icosahedron

in cosmology and the effort to describe the large-scale structure of the universe. To get a feel for how these higher dimensions are contemplated, here is another exercise in analogy. Consider the disk:

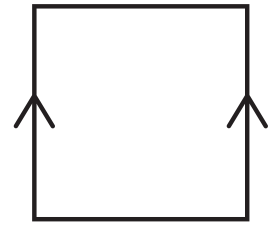


This is a two-dimensional object. If we consider this disk as viewed edge-on in space and push the center downward, we get a bowl:

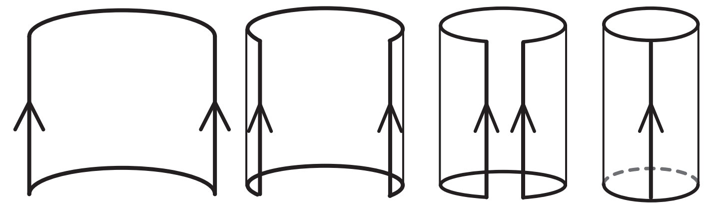
Stretching the rim of the bowl to touch a point above the bowl forms a sphere:



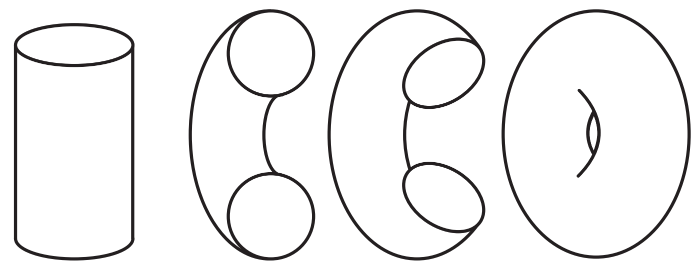
Now consider a square with the sides oriented as follows:



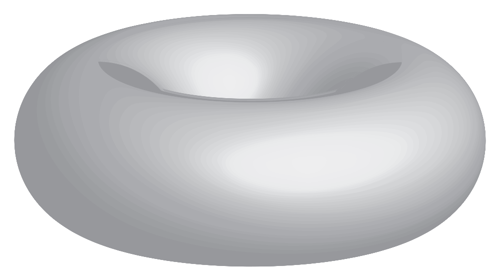
Connecting the opposing sides results in a cylinder:



Gluing the top and bottom together gives an inner tube shape:

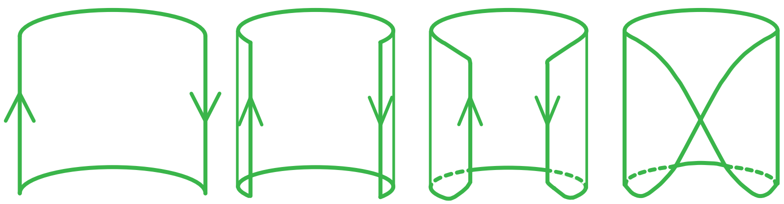


The resulting shape is called a *torus:*



Thus, by simply gluing the edges of a square together, we can get a shape that looks completely different. Notice that from the perspective of the square, traveling to one edge transports the traveler back to the opposite edge, which accounts for the two perpendicular circular directions on the torus. Imagine living as an ant on the square: every time you departed the left edge you'd appear on the right edge. The same thing happens seamlessly on the torus.

Consider next a similar square in which only the left and right sides have arrows oppositely directed. The process of identifying those sides can be seen as follows:



The resulting glued object is called a *Möbius band*. Such an object possesses the feature of being "unoriented" in that, in contrast with a cylinder, it fails to have a distinct inside and outside, so it is considered *one-sided*. This can be seen in this rendering of a Möbius band:

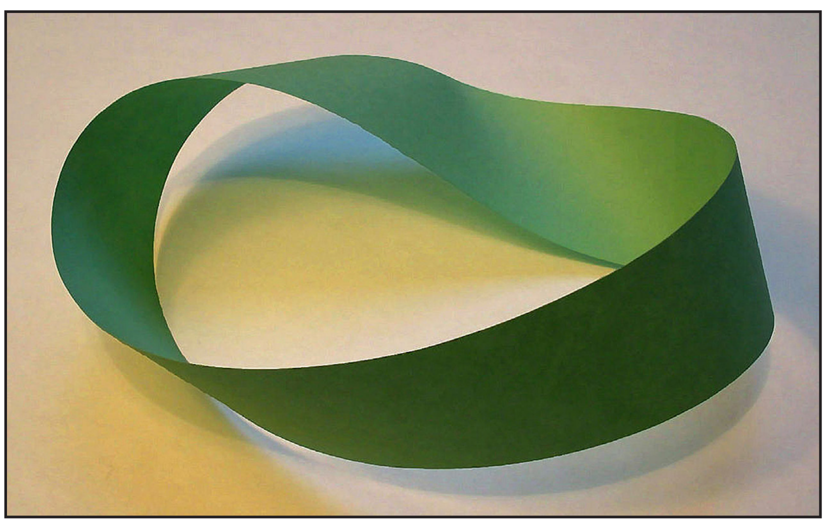
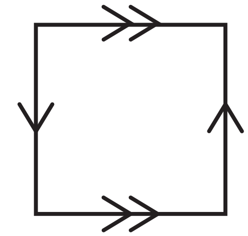


IMAGE CREDIT: WIKIPEDIA COMMONS

Finally, consider the square with edges oriented as follows:



Gluing opposite edges in the way that aligns the arrows yields an unoriented surface known as the Klein bottle:

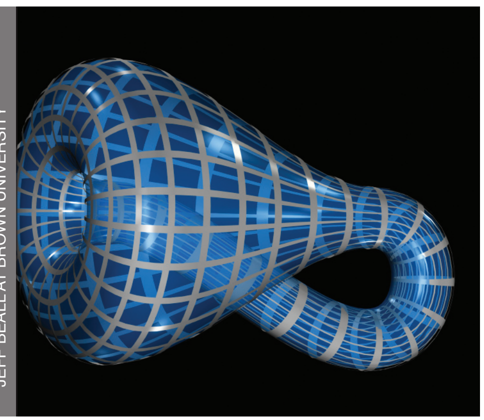


IMAGE CREDIT: THOMAS BANCHOFF AND JEFF BEALL AT BROWN UNIVERSITY

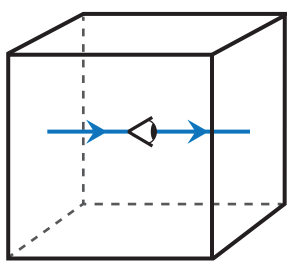
The unorientabilty of this surface prevents it from having both an inside and an outside: an ant can crawl from the inside to the outside without reaching an edge. The picture is deceptive. To view it in three dimensions, as above, requires the neck of the bottle to pierce the body. Four spatial dimensions are required to give a proper depiction. This is the sort of hidden reality that excited me as a student; simply matching up edges on a square can yield a bizarre shape that can't be portrayed in three dimensions.

We can make note of three features of the surfaces the sphere, the torus, and the Klein bottle. They are:

- **Locally two-dimensional:** Because we formed them by gluing the edges of a flat square, the surface looks flat and two-dimensional when focusing on any point up close.

- **Closed:** While the square has an edge or boundary, after gluing the edges the boundary disappears making it edgeless or closed.
- **Embedded in higher dimensions:** Even though these surfaces are two-dimensional up close, being closed forces them to have three or even four spatial dimensions.

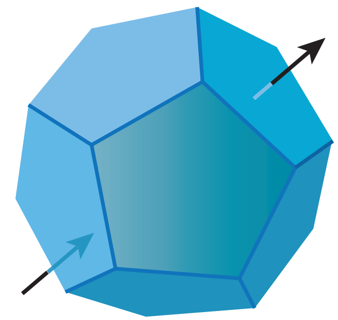
A similar view can be given to the description of our universe. Cosmologists develop models of the cosmos based upon general relativity and supported by astronomical observations. Among them there are some models that are geometrically closed. We know from our own experience that the world is locally three-dimensional. What are the possible geometric descriptions of such a closed three-dimensional structure? By analogy, instead of starting with a square and selecting rules for gluing the outer edge, start with a cube:



Imagining the interior to be our universe, we may consider opposite faces glued according to variations on the gluing rules we contemplated for the square. Performing this gluing for all three pairs of opposing faces gives a closed, locally three-dimensional object which, because of gluing all opposing faces, requires embedding in more than three spatial dimensions to properly exist. Note, as with the torus, that viewing beyond a face brings one's visual field back into the cube through the opposite side, right behind the viewer (you could see the back of your own head).

It should be noted, that other polyhedra can be considered when forming models of the universe. For example, based on astronomical

observations, Jeff Weeks has proposed in *The Shape of Space* that gluing opposing faces of dodecahedron gives a good closed model of our universe:



This geometric object is called a *Siefert-Weber manifold*.

For a mathematician, the exploration of higher dimensions need not end at four dimensions. For example, in our discussion of Platonic solids, we identified the only five that exist in three-dimensional space. In four-dimensional space, we indicated there is a hyper version of each of the five of the Platonic solids. Are there others? The definition of hyper-Platonic solids does not necessarily exclude other possibilities and, in fact, there is one more, called the *24-cell*, whose projection into three-dimensional space is displayed here:

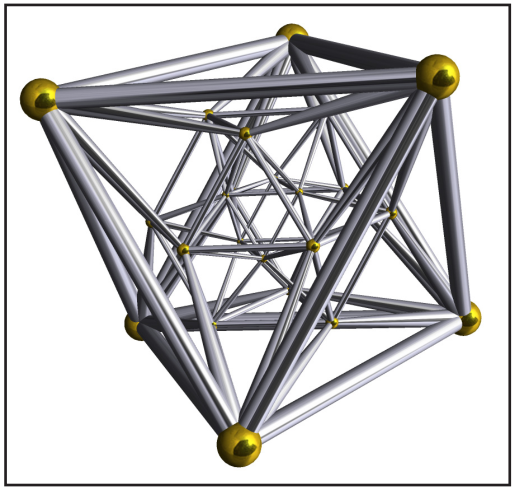


IMAGE CREDIT: WIKIPEDIA COMMONS

The definition of Platonic solids and hyper-Platonic solids can be generalized to five, six, seven, and beyond to the notion of *regular polytope*. Can a similar classification be given to such objects in such higher dimensions, or do things become overly complex? Well, the notions of hyper-tetrahedron, hyper-cube, and hyper-octahedron persist easily to every dimension. The amazing thing is that *in each dimension of five or higher, the only regular polytopes are the appropriate analogs of the hyper-tetrahedron, hyper-cube, and hyper-octahedron.* Here we have a wondrous example of the treasures that can be found within the mathematician's imagination. In the expectation that higher dimensions imply higher complexity, which in general is true, the high order symmetry of regular polytopes restricts its possibilities to just the most basic types. Unfortunately, or fortunately, geometric objects in higher dimensions generally can take on a variety of complex and exotic features for which any specific assertions that can be declared by a mathematician regarding them often require a list presuppositions in order to get a firm grasp upon their nature. However, in the case of Platonic solids, the use of analogy and symmetry as a means of discerning these geometric objects leads not only to the types of simplicity and unexpected connections that give a breathtaking beauty to our understanding of higher dimensions; it also provides a way of initiating the search for deeper connections as it relates to more general geometric objects in any dimensions by first considering them as a suitably general form of polytope.

The Splendor of Creation: Beauty and the Glory of God

By taking geometric objects and their symmetries as the source of examples of beauty in mathematics, my aim is to offer a sense of how mathematics instills a sense of awe in mathematicians and scientists as they explore the deep inner workings of physical space. Moreover, one can glean from such explorations a sense that physical space is not required to be the way it is. From the viewpoint of mathematics, there is a wealth of possibility for how space can be woven together to give a geometry

for the fabric of the cosmos. From such a vantage point, one can easily see the universe as a creation—a creation intended to produce wonder in participants with whom the Creator desires a relationship. Some of these participants may be enraptured by the equations they are contemplating, which disclose in geometric designs the impress of a divine author at the root of our entire existence. Herein is beauty found: to see the presence of the Creator revealed in the designs and relations eloquently articulated through the equations of the physicist or mathematician. Furthermore, in the expressions of the mathematician's world, the colors of creation's possibilities can be discerned to be among those on the Creator's palette, perhaps as seen before the brush has even touched the canvas.

Among the ways that beauty finds its presence within a mathematical discourse are the unexpected connections revealed within the physical makeup of reality, the pleasing encounter with and fruitful productivity from symmetric relations, and the contemplation of transcendent realities within higher dimensions surmised through the power of analogy. Each of these can elicit awe from the mathematician, the scientist, the pastor, and the parishioner alike as they examine the nature of space in its geometric forms. That we may contemplate the ways reality both *is* and *could be* is a source of great mystery. If one is willing to step back to take it all in, it can inspire a sense of awe and a consideration of the possibility of a divine author to all that there is—perhaps leading the one contemplating to respond in the most profound fashion: Glory!

Further Reading

Beauty in mathematics and science

- Huntley, H.E. *Divine Proportion*. Dover Publications, 1970.
- Sinclair, N., D. Pimm and W. Higginson, eds. *Mathematics and the Aesthetic: A New Approach to an Ancient Affinity*. New York: Springer-Verlag, 2006.
- Chandrasekhar, S. *Truth and Beauty: Aesthetic Motivations in Science*. Chicago: University of Chicago Press, 1990.

Symmetry

- Stewart, Ian. *Why Beauty is Truth: A History of Symmetry*. New York: Basic Books, 2007.

Geometry, polyhedra, and higher dimensions

- Richeson, David S. *Euler's Gem*. Princeton: Princeton University Press, 2008.
- Weeks, Jeff. *Shape of Space*, 2nd ed. CRC Press, 2001.
- Abbott, Edwin Abbott. *Flatland: a Romance of Many Dimensions*. Dover Publications, 1992. A film version of *Flatland* is available at http://www.flatlandthemovie.com.

Beauty and theology

- Farley, Edward. *Faith and Beauty: A Theological Aesthetic*. London: Ashgate Publishing, 2001
- Hart, David Bentley. *The Beauty of the Infinite*. Grand Rapids, MI: Wm. B. Eerdmans Publishing, 2004.
- Sherry, Patrick. *Spirit and Beauty*, 2nd ed. London: SCM Press, 2010.